Feedback and Fluctuations in a Totally Asymmetric Simple Exclusion Process with Finite Resources

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Abstract. We revisit a totally asymmetric simple exclusion process (TASEP) with open boundaries and a global constraint on the total number of particles [Adams, et. al. 2008 *J. Stat. Mech.* P06009]. In this model, the entry rate of particles into the lattice depends on the number available in the reservoir. Thus, the total occupation on the lattice feeds back into its filling process. Although a simple domain wall theory provided reasonably good predictions for Monte Carlo simulation results for certain quantities, it did not account for the fluctuations of this feedback. We generalize the previous study and find dramatically improved predictions for, e.g., the density profile on the lattice and provide a better understanding of the phenomenon of "shock localization."

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1. Introduction

The Totally Asymmetric Simple Exclusion Process (TASEP) is a simple, yet rich model in the poorly understood and vast realm of non-equilibrium statistical mechanics. Particles are placed in a hypercubic lattice (for example) and hop randomly to a nearest neighbor empty site, *except* along one of the axes where the hopping is *uni-directional*. Since its dynamics violate detailed balance, its stationary states are non-trivial, with typically no equivalence to any states in equilibrium statistical mechanics. For a system with periodic boundary conditions, the stationary state is trivial, with all configurations having equal probability (i.e., a flat distribution)[1]. However, the dynamic properties are far from trivial, being quite distinct from those for a completely symmetric exclusion processes (i.e., simple diffusion)[2]. For systems with open boundaries, in which particles hop in and out of the system with various rates, even the stationary states are quite complex. Indeed, an open TASEP in just one dimension settles into three different phases, depending on the entry and exit rates[3]. Despite the simplicity of its microscopic rules of evolution, the solution to this process was not known analytically until recently[4, 5, 6]. Needless to say, its dynamic properties are even more complex[7, 8, 9, 10]. Examples of comprehensive reviews on this "simple" process include [11, 12]. Meanwhile, extensions of such a one-dimensional TASEP are of great

interest, since they may be applied to real systems such as protein synthesis [13], biomolecular motors [15, 16], traffic flow [14], and surface growth [17].

More recently, there are two studies involving a special generalization of the open TASEP. Here, the reservoir, or "pool," from which particles are injected into the lattice is *finite*. In particular, the total number of particles on the lattice, N, plus the number in the pool, N_p, is a fixed constant: N_{tot}. Such a constraint can be understood in the context of protein synthesis as having a finite number of ribosomes (which model particles in TASEP) in a cell[18], or in the context of traffic (with cars as particles) as the "parking garage problem" [19]. The two studies differ in how particles are moved from the reservoir into the lattice, resulting in quite different behavior. In particular, the former investigated the properties of N and the average particle current, J, when N_{tot} is varied. While $J\left(N_{tot}\right)$ displays no surprises, $N\left(N_{tot}\right)$ shows some remarkable features (when the entry/exit rates are chosen so that, in the $N_{tot} \to \infty$ limit, the system is in a high-density phase or the "shock" phase). Using simple arguments of self-consistency, as well as more sophisticated domain wall (DW) theory[8, 9], many of these phenomena can be reproduced [18]. In this paper, we extend this investigation in significant ways and provide better insight into the effects of imposing a fixed N_{tot} on TASEP. In particular, we show that, although the previous theory for $N\left(N_{tot}\right)$ appears quite adequate everywhere, the agreement is deceptive in certain cases. Our improvements are not merely incremental; they provide a full understanding of the phenomenon of "shock localization." As a consequence, our prediction of the average density profile, which is entirely different from that of the simple DW theory, is in excellent agreement with simulation results.

We should note that shock localization has been observed previously [15]. However, the underlying mechanisms are distinct. In particular, the earlier studies focus on TASEP's with Langmuir kinetics, i.e., one with no particle conservation. To model biomoluecular motors, which can attach and detach from a microtubule, it is natural to let particles appear and disappear with various rates along the entire lattice. As a result, in approaches that use DW theory and shock dynamics [16], the shock is understood to be localized by a subtle interplay between adsorption and desorption of the particles. By contrast, our study here involves a TASEP with particle conservation, not only within the lattice, but also including the reservoir. The shock is localized through a much simpler mechanism: the interplay between the lattice and the reservoir. As will be shown below, the mathematical methods used are quite similiar: site dependent hopping rates for the domain wall.

This paper is organized as follows. In the next section, we provide the details of the model and summarize previous results. Section 3 is devoted to both new Monte Carlo data and the improved DW theory. A summary and outlook for future studies form the concluding section.

2. Model definition and previous findings

The standard open TASEP consists of a one-dimensional lattice of sites labeled by $i = 1, ..., L.$, each of which can be vacant or occupied by a single particle. Thus, the configurations can be specified by the set of occupation numbers $\{n_i\}$, with $n_i = 0, 1$ being the particle content of site i. Thus, $N = \Sigma_i n_i$. The (random sequential) dynamics is implemented by choosing a particle at random and moving it to the next site with unit rate, provided the target site is not occupied. For a particle at site L (the right boundary), it leaves the system with rate β. At the left boundary, a particle can enter the system at site 1 with rate α. When the system reaches a stationary state, the average overall density

$$\rho \equiv \langle N \rangle / L$$

will be constant. As we vary α and β, the system can be found in three different phases: high density (HD), low density (LD), and maximal current (MC). The MC phase prevails if both α and β are $\geq 1/2$. In the thermodynamic limit, ρ is $1/2$ and J, the average current, is $1/4$, regardless of (α,β). If $\beta < 1/2$ and $\alpha > \beta$, the system settles into a HD phase, with $\rho = 1 - \beta$ and $J = \beta(1 - \beta)$. Due to particle-hole symmetry, the LD phase is similar, with $\rho = \alpha$ and $J = \alpha(1 - \alpha)$. for $\alpha < 1/2$ and $\beta > \alpha$. The transitions across the phase boundaries HD-MC and LD-MC are continuous. On the line $\alpha = \beta$, a HD region coexists with a LD one, separated by a microscopic interface, known as the shock (a term aptly describing a car driving from a high-speed, low-density region into a traffic jam). The position of the shock wanders due to fluctuations, so that the long-time *average* of ρ is again $1/2$. Systems displaying such co-existence are often referred to as being in the "shock phase" (SP). A concise summary for standard TASEP is

$$\rho_\infty(\alpha, \beta) = \begin{cases} 1/2 & \text{MC; SP} & \alpha, \beta \geq 1/2;\ \alpha = \beta \leq 1/2 \\ \alpha & \text{LD} & \alpha \leq \min(\beta, 1/2) \\ 1 - \beta & \text{HD} & \beta \leq \min(\alpha, 1/2) \end{cases} \tag{1}$$

where we have used the subscript ∞ to remind the reader of its relationship to our constrained system ($N_{tot} \to \infty$). The expression for the average current is, in all cases, $\rho_\infty(1 - \rho_\infty)$.

In the constrained TASEP, the entry-rate is chosen to depend on N_p, the number of particles in the reservoir. Denoting this rate by α_{eff}, we characterize such a dependence through a function f:

$$\alpha_{eff} = \alpha f(N_p). \tag{2}$$

The advantage of this form is that, by choosing $f \to 1$ in the limit of large argument, we will recover the standard TASEP with the parameters (α,β) when the total number of particles in the system

$$N_{tot} = N_p + N \tag{3}$$

is unlimited. In [19], the reservoir models a "parking garage" and the lattice models a road, so that f is chosen to be the simplest function with the desired asymptotic

property: $f(N_p > 0) = 1$. Of course, $f(0) = 0$, since no car can emerge from an empty garage. By contrast, the motivation in [18] is the modeling of initiation (a ribosome attaching onto an mRNA, to begin the process of protein synthesis) that is limited by the ribosome concentration in the cell. Thus, they chose $f(x) \propto x$, for small x. Much of that study was based on the specific function

$$f(N_p) = \tanh(N_p/N^*) \tag{4}$$

where N^* models some cross-over level. As a simple starting point, it was chosen to be the average number of particles in the standard TASEP, i.e., $N^* = \rho_\infty(\alpha, \beta) L$. Thus,

$$N^* = \begin{cases} L/2 & \alpha, \beta \geq 1/2;\ \alpha = \beta \leq 1/2 \\ \alpha L & \alpha \leq \min(\beta, 1/2) \\ (1-\beta) L & \beta \leq \min(\alpha, 1/2) \end{cases} \tag{5}$$

associated with the MC, LD, and HD states, respectively. As a result, it is more convenient to express this function in terms of an intensive control parameter

$$\rho_{tot} \equiv N_{tot}/L \tag{6}$$

and regard f as a function of ρ. For example, for the LD case, we have $\alpha_{eff} = \alpha \tanh[(\rho_{tot} - \rho)/\alpha]$. The main focus in [18] are $\rho(\rho_{tot}; \alpha, \beta)$ and $J(\rho_{tot}; \alpha, \beta)$, with the knowledge that $\rho(\infty; \alpha, \beta) = \rho_\infty(\alpha, \beta)$ of Eqn. (1). Since predictions for J follow those for ρ, it is sufficient to focus only on the latter. In any case, the variations in J are relatively minimal and not as spectacular as those in ρ.

To appreciate the different phenomena displayed by the constrained TASEP, it is important to keep in mind that the effective entry-rate varies from 0 to α, as ρ_{tot} is increased from 0 to ∞. Thus, if $\alpha \ll \min(\beta, 1/2)$ (i.e., the "LD case"), then phase boundaries are neither traversed nor approached asymptotically. On the other hand, if the (α,β) are chosen so that the unconstrained TASEP is deep in the HD or MC phase, we will cross a phase boundary, so that $\rho(\rho_{tot})$ is expected to display two branches. In the "HD case", there is a third branch, which reflects the presence of coexistence, similar to the dependence of pressure on specific volume, for a liquid-vapour system below the critical point. In this branch, $\alpha_{eff} \cong \beta$ while ρ is essentially linear in ρ_{tot}: $\rho \cong \alpha\rho_{tot}/(1 - \beta + \alpha)$. In general, the finite size effects, by the time L reaches 1000, are hardly noticeable. Using simple self-consistency arguments, the predicted $\rho(\rho_{tot})$ is in reasonably good agreement with simulation data [18]. For example, setting ρ to α_{eff} for the "LD case" (in accordance with Eqn. (1)), we have $\rho = \alpha \tanh[(\rho_{tot} - \rho)/\alpha]$. This is a transcendental equation for ρ, much like the one for magnetisation as a function of the external field in the "zeroth" approximation of the Ising model, and leads to $\rho(\rho_{tot}; \alpha)$. The most challenging case is "SP," both for simulations and analytic understanding. First, the system takes a long time to settle into stationary states. Second, $\rho(\rho_{tot})$ shows *two* cross-overs before saturating at 1/2. Furthermore, the location of the second cross-over depends strongly on L. Meanwhile, DW theory for the standard TASEP [9] accounts for some effects of finite L and provides a more detailed prediction than Eqn. (1). We denote this result of DW theory by $\rho_{DW}(\alpha, \beta, L)$. Replacing α in this

formula by $\alpha_{eff} = \alpha \tanh[2(\rho_{tot} - \rho)]$, another implicit equation for ρ is established [18]. Solving such an equation leads to a $\rho(\rho_{tot})$ that contains all the surprising features observed. Indeed, the predicted $\rho(\rho_{tot})$ agrees very well with the data ($\alpha = \beta = 0.25$ and $L = 1000$) everywhere, except in the middle of the second cross-over, where the worst disagreement is about 6% [18]. Unpublished data for $\alpha = \beta = 0.25$ and $L = 100$ and analytic results of $\rho(\rho_{tot})$ showed [20] that this second cross-over essentially vanished, while the maximum disagreement there between theory and simulations is about 7%. In Fig. 1, the solid orange symbols represent these two sets of data, while the two lines are predictions – with no fitting parameters – from this "simple" domain wall (SDW) theory. Given such remarkably good agreements, it may seem an overkill to pursue this problem further. However, as we will see in the next section, the agreement in [18] is *deceptively* good. Had a more sensitive quantity like the average density *profile* ($\rho_i \equiv \langle n_i \rangle$) been considered, a glaring discrepancy would have been exposed.

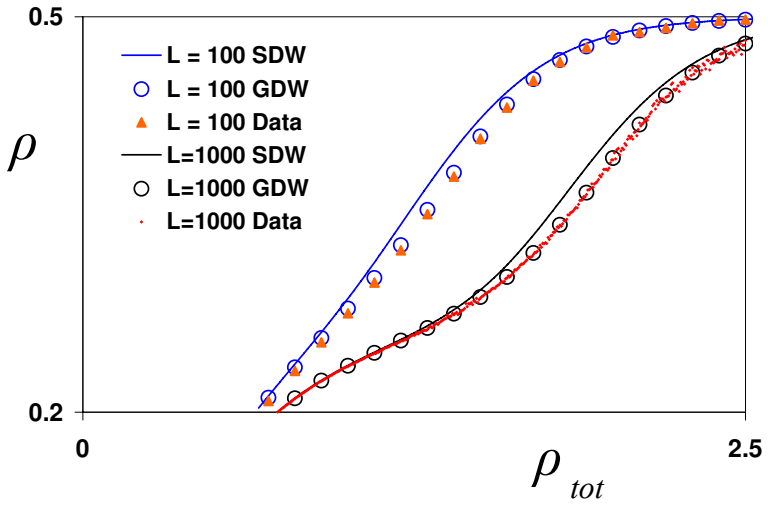

Figure 1. Average overall density as a function of N_{tot} for L=100 and L=1000 with $\alpha = \beta = 0.25$.

3. Domain wall theory for constrained TASEP

The domain wall (DW) theory used in [18] is formulated for a *constant* entry (and exit) rate [9], appropriate for the standard, unconstrained TASEP. Referring the reader to the details in [9], let us summarize the key points here. The central premise of this theory is to let the configurations of the system be approximated by two regions of low/high densities, separated by an interface (i.e., DW) of zero "intrinsic width." To be specific, a region of low (local) density, ρ_- $(= \alpha)$, on sites up to k, is connected to a region of high density, ρ_+ $(= 1 - \beta)$, on sites $\in [k+1, L]$, so that only a single integer (k) is used to label each configuration. The dynamics of the DW is implemented through its drifting rates to the right/left, $D_\pm$, dictated by both the densities on either side of the shock and the particle fluxes from the two open ends, $j_\pm$. The final result is a master

equation for the probability to find the DW at site k and time t, $P(k,t)$:

$$\partial_t P(k,t) = D_+ P(k-1,t) + D_- P(k+1,t) - (D_+ + D_-) P(k,t) \tag{7}$$

in the bulk ($k \in [1, L-1]$), with

$$D_+ = \frac{j_+}{\rho_+ - \rho_-} = \frac{\beta(1-\beta)}{1-\beta-\alpha} \tag{8}$$

$$D_- = \frac{j_-}{\rho_+ - \rho_-} = \frac{\alpha(1-\alpha)}{1-\beta-\alpha} \tag{9}$$

When the DW gets to the boundaries, it reflects back into the system, so that

$$\partial_t P(0,t) = D_- P(1,t) - D_+ P(0,t) \tag{10}$$

$$\partial_t P(L,t) = D_+ P(L-1,t) - D_- P(L,t) \tag{11}$$

and $P(k<0,t) = P(k>L,t) \equiv 0$. The stationary solution is a simple exponential:

$$P^*(k) = r^{-k}/\mathcal{N} \tag{12}$$

where

$$r \equiv \frac{D_-}{D_+} = \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \tag{13}$$

and $\mathcal{N}$ is a normalization factor. It is clear that, for the HD/LD phases, r is greater/lesser than unity and the DW is localized at the left/right end of the lattice. On the other hand, r is unity for SP, so that the DW can be found anywhere on the lattice.

Turning to our problem of a constrained TASEP, we recognize that, especially if ρ_{tot} is not large, the feedback from the lattice occupation cannot be neglected. As indicated above, this is incorporated in the previous study [18] by introducing an $\alpha_{eff}(\rho)$ and solving a self consistent equation for ρ. But this ρ is just the average overall density, so that the changes in α_{eff} during a simulation run are completely unaccounted for. Here, we wish to investigate how important are the density fluctuations, given that they feed back into α_{eff}. To distinguish the fluctuating entry rate from the *constant* in the SDW theory, let us denote the latter by α_{eff}^{SDW}, so that we have explicitly,

$$P^*_{SDW}(k) \propto r_{eff}^{-k}\,; \qquad r_{eff} = \alpha_{eff}^{SDW}\left(1-\alpha_{eff}^{SDW}\right)/\,\beta(1-\beta)\ . \tag{14}$$

Now, to account for how this feedback affects the diffusion of the shock in full is non-trivial. However, the essentials of the physics are clear: The feedback stabilizes the DW, even in the vicinity of SP: If the DW wanders too far to the right (larger k), there will be more particles in the pool, resulting in a higher entry-rate. This in turn enhances the flux j_- and drives the DW to the left (smaller k). A similar stabilizing action occurs if the DW wanders too far to the left. To implement this idea and formulate a "generalized" domain wall (GDW) theory, we will make a drastic approximation (neglecting time delays, etc.), consisting of three ingredients:

(i) replacing α by α_{eff} everywhere

(ii) substituting $\rho = \rho_-(k/L) + \rho_+(1-k/L)$ into $\alpha_{eff}(\rho)$

(iii) letting ρ_- be α_{eff} and obtaining a k-dependent $\alpha_{eff,k}$ through a self-consistent equation.

The last point is subtle and deserves clarification. As $\alpha_{eff} = \alpha f(N_p)$, i.e., $\alpha \tanh[(\rho_{tot} - \rho)/\alpha]$ here, we have an implicit equation for determining $\alpha_{eff,k}$:

$$\alpha_{eff} = \alpha \tanh\left[\left(\rho_{tot} - \rho_-(k/L) - \rho_+(1 - k/L)\right)/\alpha\right] \quad \Rightarrow \tag{15}$$

$$\alpha_{eff,k} = \alpha \tanh\left[\left(\rho_{tot} - \alpha_{eff,k}(k/L) - (1-\beta)(1 - k/L)\right)/\alpha\right] \tag{16}$$

Once $\alpha_{eff,k}$ is found, it will enter in defining the k-dependent drift rates:

$$D_{+,k} = \frac{\beta(1-\beta)}{1 - \beta - \alpha_{eff,k}} \tag{17}$$

$$D_{-,k} = \frac{\alpha_{eff,k}(1 - \alpha_{eff,k})}{1 - \beta - \alpha_{eff,k}}. \tag{18}$$

The result of these considerations is a generalized master equation for $P(k,t)$ which, in the "bulk," reads

$$\partial_t P(k) = D_{+,k-1}P(k-1) + D_{-,k+1}P(k+1) - (D_{+,k} + D_{-,k})P(k) \tag{19}$$

(with the variable t suppressed). For the boundary equations, there is a further complication which we must account for. When the resources are so limited that the lattice cannot be fully filled at density ρ_+, i.e., when $N_{tot} < \rho_+ L$, the DW cannot be located arbitrarily far to the left. Thus, the interval available to k is $[k_{\min}, L]$, where

$$k_{\min} = L - \frac{N_{tot}}{1-\beta}. \tag{20}$$

Note that $k = k_{\min}$ corresponds to the configuration with $N_p = 0$, and so, $\rho_- = \alpha_{eff}(0) = 0$. With this limitation in mind, we obtain the boundary terms for the master equation:

$$\partial_t P(k_{\min}) = D_{-,k_{\min}+1}P(k_{\min}+1) - D_{+,k_{\min}}P(k_{\min}) \tag{21}$$

$$\partial_t P(L) = D_{+,L-1}P(L-1) - D_{-,L}P(L) \tag{22}$$

Despite the extra complications, the stationary distribution can be found analytically[16]. Since the configuration space is one-dimensional, we have, in general, a recursion relation $P^*(k) = [D_{-,k+1}/D_{+,k}]\,P^*(k+1)$. Thus, we arrive at the solution for our constrained TASEP:

$$P^*(k) \propto \prod_{l=k}^{L-1} \frac{D_{-,\ell+1}}{D_{+,\ell}} \tag{23}$$

and, with proper normalization,

$$P^*(k) = \left(\sum_{m=k_{\min}}^{k-1} \prod_{\ell=m+1}^{k} \frac{D_{-,\ell}}{D_{+,\ell-1}} + 1 + \sum_{m=k+1}^{L} \prod_{\ell=k+1}^{m} \frac{D_{+,\ell-1}}{D_{-,\ell}} \right)^{-1} \tag{24}$$

Since this $P^*(k)$ carries information on whether the DW is located before or after a given site i, we can compute the average density profile:

$$\rho_i = \sum_{k=k_{\min}}^{i} (1-\beta)P^*(k) + \sum_{k=i+1}^{L} \alpha_{eff,k}P^*(k) \tag{25}$$

and of course, the overall density is given by $\rho = \sum \rho_i / L$. Needless to say, these results are quite different from those in the simple DW theory, e.g., Eqn. (12).

To highlight how differently the two predictions compare with simulation data, we choose just one point in parameter space – a point in the "third branch" of $\rho\,(\rho_{tot})$ in the HD case. Here, the essential physics revealed by simulations is that the number of particles in the pool remains more or less constant as N_{tot} is increased, with the extra particles being absorbed by the lattice to form a high density region near the exit. Furthermore, the shock on the lattice is seen to be quite localized, as revealed by the average density profile. Fig. 2 shows this behavior for the case of $L = 1000$, $N_{tot} = 800$, $\alpha = 0.75$, and $\beta = 0.25$. In this figure, we also plot the predictions from the two theories, Eqns. (12,24). Though both provide quite good agreement with the overall density (area under the profile), it is clear that shock localization cannot be achieved by using an exponential $P^*(k)$. By contrast, accounting for the feedback through site-dependence drift rates for the DW[16], the shock is localized. Given how crude our approximations are, it is remarkable that the agreement with data is so good. We conclude that even this simple-minded level of accounting for feedback can capture the essence of shock localization.

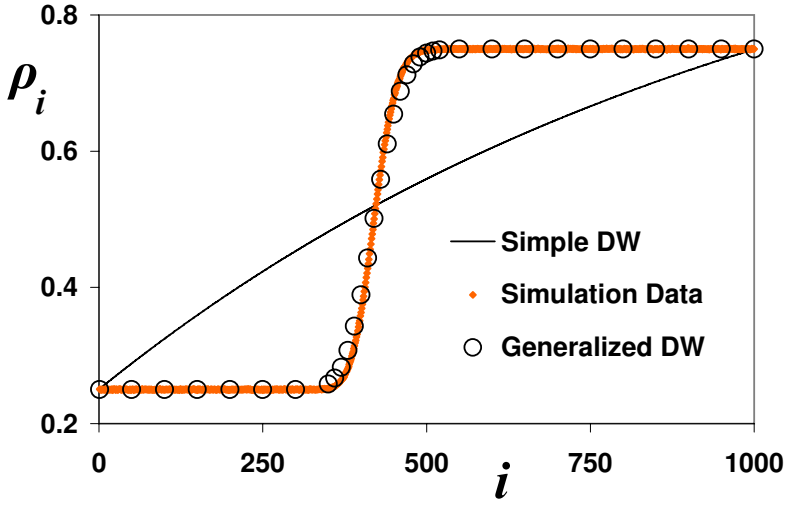

Figure 2. Average density profile for $L = 1000$ with $N_{tot} = 800$, $\alpha = 0.75$, and $\beta = 0.25$.

Apart from the dramatic improvements to the profile here, our GDW theory also provided better fits to both $\rho\,(\rho_{tot})$ and the profile in the SP case. Since our theory accounts for some feedback, we are not surprised that it is more successful at dealing with the large fluctuations associated with the SP case. Carrying out the computations detailed above, we find results that are in surprisingly good agreement with simulation data, as the open circles in Figs. 1 and 3 show. While we show two cases in the former ($L = 1000$ and $L = 100$), the latter contains only the profile for one case – $\rho_{tot} = 2$ – in the $L = 1000$ samples. Although the improvements over SDW here are not as

spectacular as in the HD case, they are still quite substantial.

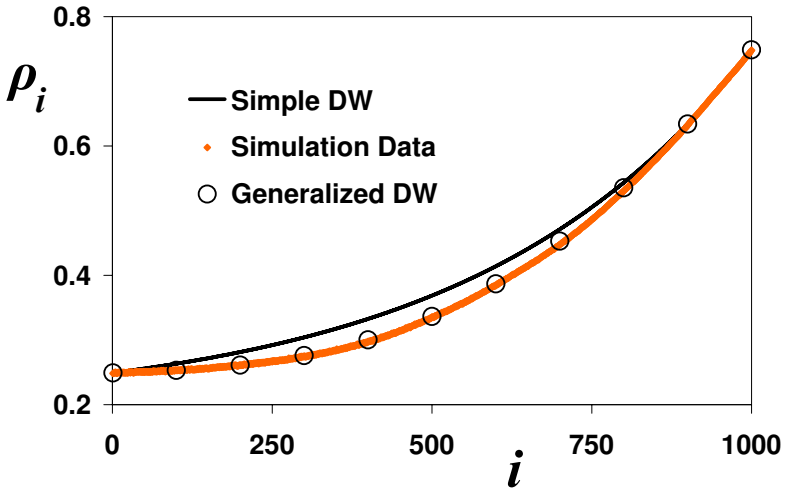

Figure 3. Average density profile for $L = 1000$ with $N_{tot} = 2000$, $\alpha = \beta = 0.25$.

Another useful perspective on the difference between the SDW and GDW theories is the following. Consider the gradient of the density profile (which is intimately related to $P^*(k)$, of course). For a discrete lattice, this is

$$\Delta\rho_k \equiv \rho_k - \rho_{k-1} \ .$$

In SDW, it is simply $(1-\beta-\alpha_{eff}^{SDW})P^*_{SDW}(k)$, and so, is a pure exponential, since α_{eff}^{SDW} is a constant. In other words, $\ln\Delta\rho_k$ is linear in k (with coefficient $\ln r_{eff}$). By contrast, in GDW, an extra k dependence appears through $\alpha_{eff,k}$ in $(1-\beta-\alpha_{eff,k})P^*(k)$. From Fig. 2, it is clear that the non-linear terms are significant and $\Delta\rho_k$ is close to a Gaussian. On the other hand, it is also clear that Fig. 3 shows that $\ln\Delta\rho_k$ is essentially linear. That GDW can account for such sharp differences is remarkable. At a deeper level, to *understand* how a simple k dependence in $\alpha_{eff,k}$ can be so successful will require a thorough analysis of the detailed properties of $f\,(N_p)$. Beyond the scope of this paper, this study is being undertaken and will be reported elsewhere.

4. Summary and outlook

In this paper, we revisit the constrained totally asymmetric simple exclusion process proposed in [18]. Instead of particles entering/exiting the lattice from/to an infinite pool, this system has a fixed and finite total number of particles, N_{tot} – the sum of those on the lattice (N) and in the pool (N_p). Further, the entry rate α_{eff} is assumed to be a simple function of N_p, in such a way that $\alpha_{eff} \propto N_p$ if the pool has few particles, yet saturating at a constant α for $N_p \to \infty$. The main interest is how N varies as N_{tot} is increased, when the exit rate is always fixed at β. In [18], Monte Carlo simulation data showed that $N\,(N_{tot};\alpha,\beta)$ displays a variety of properties, depending on the parameters (α,β). A simple domain wall theory was shown to provide good

predictions (*zero* adjustable parameters) for all the data. However, there were lingering doubts for the rationale behind such a simple theory. Here, we undertake to improve upon these theoretical considerations.

By taking into account the fluctuating feedback into the entry rates from the total particle occupation on the lattice, we formulated a "generalized" domain wall theory, in which the hopping rates for the domain wall depend on the location of the wall (similar to the spirit, but differing in the physics of earlier studies [16]). Although this approach is still somewhat heuristic, its predictions for $N\left(N_{tot};\alpha,\beta\right)$ are better than those in [18]. Furthermore, we find that the agreement in [18] turns out to be *deceptively* good for certain cases. In such situations, the previous predictions for a more detailed quantity – the density profile – can be seriously erroneous. By contrast, the generalized domain wall theory provides excellent results for the profiles in all cases. In physically understandable terms, the feedback mechanism serves to localize the domain wall. While this feedback plays a small role in most cases, its effects are all-important in other cases, with the most dramatic example displayed in Fig. 2. Our conclusion is that even a naive inclusion of the feedback improves significantly our understanding of the fluctuations in a constrained TASEP. Of course, further progress can be made, such as a more systematic approach to accounting for all the fluctuations in this system. In particular, the quantitative aspects of this feedback are clearly associated with the details of f, specifically, its derivatives. Beyond static properties, there are undoubtedly many interesting dynamic phenomena yet to be discovered. For example, we are aware of non-trivial effects[21] induced by the constraint on the power spectrum associated with the time trace of N [10] – a phenomenon that may be understood through an appropriate extension of this theory.

Finally, we should remark that one of the motivations for studying TASEP with finite resources comes from the potential applications to protein synthesis in a cell. In a real biological system, the supply of ribosomes (which are modeled by particles in TASEP) is finite. In case this supply is low, there may be observable consequences for translation. For that application, the above TASEP needs to be generalized, to include exclusion at a distance (particles covering more than one site) and inhomogeneous hopping rates (associated with a non-trivial sequence of codons) [13]. Furthermore, there are many different genes, as well as many copies of each. Thus, we would face a system of multiple copies of TASEP's, with different lengths and hopping rates. How these systems are affected by finite resources will pose many interesting new challenges, especially on the theory front. It is clear that our study here is but a very small step towards the understanding of protein synthesis *in vivo*.

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